Multivariate Functions

Goal: Familiarize ourselves with functions of several variables.

A multivariate function is a function depending on multiple variables.
More formally, a real-valued function of multiple variables is a function $f: D \to \mathbb{R}$ with domain $\mathrm{dom}(f) = D \subseteq \mathbb{R}^n$.
The range of such a function is $\mathrm{ran}(f) = \{f(x) : x \in \mathrm{dom}(f)\} \subseteq \mathbb{R}$

Ex: Here are some 2-variable functions:

1. $f(x,y) = xy$
2. $g(x,y) = \sqrt{1-x^2-y^2}$
3. $h(x,y) = \dfrac{x-y}{x^3+y^3}$
4. $\alpha(x,y) = \cos(x) + \sin(y)$
5. $\beta(x,y,z) = \dfrac{x+3y-7z}{\cos(xy)}$ ← 3-variable...

Function evaluation works the same as before:

Ex: $f(3,2) = 3 \cdot 2 = 6$

$g(\tfrac{1}{2}, \tfrac{1}{2}) = \sqrt{1-(\tfrac{1}{2})^2-(\tfrac{1}{2})^2} = \sqrt{1-\tfrac{1}{4}-\tfrac{1}{4}} = \sqrt{\tfrac{1}{2}}$

$h(1,2) = \dfrac{1-2}{1^3+2^3} = -\dfrac{1}{9}$

Ex: Compute the (natural) domain of $g(x,y) = \sqrt{1-x^2-y^2}$.

Sol: $g(x,y)$ is defined as long as $1-x^2-y^2 \geq 0$

i.e. $1 \geq x^2+y^2$ i.e. $\mathrm{dom}(g) = \{(x,y) : x^2+y^2 \leq 1\}$



Multivariate functions also have graphs, although they live in higher dimensional spaces. Formally, for a multivariate function f of ~~n~~ 2 variables, we have

$$\text{graph}(f) = \{(\vec{x}, f(\vec{x})) : \vec{x} \in \text{dom}(f)\} \subseteq \mathbb{R}^{2+1}.$$

(input: $\vec{x}$; output: $f(\vec{x})$)

Ex: Describe the graph of $f(x,y) = 6x + 3y - 4$.

Sol: $z = f(x,y) = 6x + 3y - 4$

$\iff -6x - 3y + z = -4 \iff \langle -6, -3, 1\rangle \cdot \langle x, y, z\rangle = -4$ (Plane)

Hence graph(f) is a plane in $\mathbb{R}^3$. ☒

Ex: Describe the graph of $f(x,y) = \sqrt{1 - x^2 - y^2}$.

Sol: $z = f(x,y) = \sqrt{1-x^2-y^2}$

$\implies z^2 = 1 - x^2 - y^2 \iff x^2 + y^2 + z^2 = 1$ (unit sphere centered at the origin.)

NB: the unit sphere is NOT the graph of a function! b/c $(0,0)$ has two associated "outputs" i.e. $(0,0,1)$ and ~~$(0,0,-1)$~~ are both on the unit sphere...

$$\text{graph}(f) = \{(x,y,z) : z = \sqrt{1-x^2-y^2}\}$$
$$= \{(x,y,z) : z \geq 0 \text{ and } x^2+y^2+z^2 = 1\}$$

Is the upper hemisphere of the unit sphere. ☒

Ex: Use an online function grapher to investigate the following:

① $f(x,y) = xy$ ② $g(x,y) = \frac{x-y}{x^3+y^3}$ ③ $h(x,y) = \cos(x) + \sin(y)$

Often it is helpful to study a function using its <u>level curves</u>. These are (essentially) intersections of the graph with a plane. $z = k$

The <u>level curve</u> of $f(x,y)$ at <u>$z=k$</u> is the "curve" in the xy-plane defined implicitly by $f(x,y) = k$.

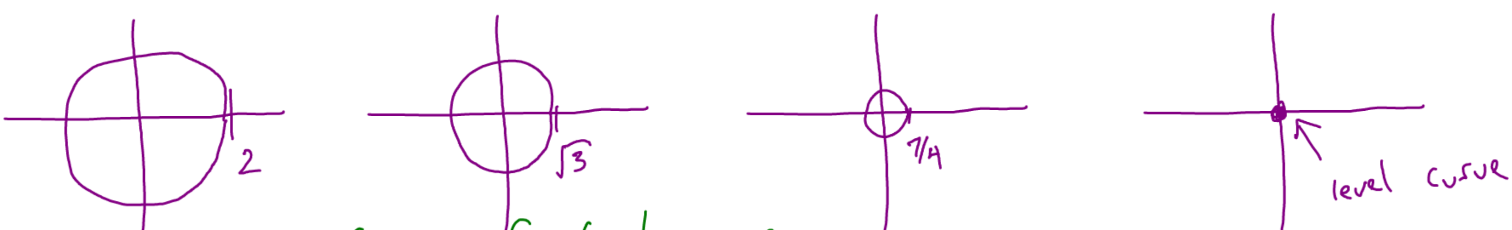
<u>Ex</u>: Describe the cross sections of $f(x,y) = \sqrt{4 - x^2 - y^2}$.

<u>Sol</u>: $k = \sqrt{4-x^2-y^2} \implies x^2+y^2 = 4-k^2$

in the plane $z=k$ yields a circle of radius $\sqrt{4-k^2}$.

<u>Sketch of level curves</u>:

$k=0$ $\quad$ $k=1$ $\quad$ $k=3/2$ $\quad$ $k=2$



A <u>Contour map</u> of $f(x,y)$ is an arrangement of level curves graphed together in the plane (Think of a <u>mountain map</u>).

→ synonym for level curve

<u>Ex</u>: Make a contour map of $f(x,y) = \sqrt{4-x^2-y^2}$.

